Section 1.1.1

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Field: Applied algebraic geometry
Key Results: To define a total polarization map and a kth polar hypersurface.

1. The Polar Pairing

Let E be the vector space over $\mathbb{C}$, and $E^\vee$ its dual vector space. Let $S^d(E)$ be a space of homogeneous polynomial of degree d over E.

1.1 Polar pairing

A polar pairing is the map

$$\langle \cdot, \cdot \rangle : S^d(E^\vee) \otimes S^d(E) \to \mathbb{C}, \tag{1}$$

which is defined by

$$\langle \ell_1 \ell_2 \dots \ell_d, w_1 w_2 \dots w_d \rangle = \sum_{\sigma \in \mathfrak{S}} \ell_{\sigma^{-1}(1)}(w_1) \ell_{\sigma^{-1}(2)}(w_2) \dots \ell_{\sigma^{-1}(d)}(w_d) \tag{2}$$

For example,

$$\langle \ell_1 \ell_2, w_1 w_2 \rangle = \ell_1(w_1)\ell_2(w_2) + \ell_2(w_1)\ell_1(w_2).$$

1.2 A partial polarization map

A polar pairing is the map

$$\langle \cdot, \cdot \rangle : S^d(E^\vee) \otimes S^k(E) \to S^{d-k}(E^\vee), \quad k \le d \tag{3}$$

which is defined by

$$\langle \ell_1 \ell_2 \dots \ell_d, w_1 w_2 \dots w_k \rangle = \sum_{1 \le i_1 \le \dots \le i_k \le n} \langle \ell_1 \ell_2 \dots \ell_k, w_1 w_2 \dots w_k \rangle \prod_{j \ne i_1, \dots, i_k} \ell_j. \tag{4}$$

For example,

$$\langle \ell_1 \ell_2 \ell_3, w_1 w_2 \rangle = \langle \ell_1 \ell_2, w_1 w_2 \rangle \ell_3 + \langle \ell_1 \ell_3, w_1 w_2 \rangle \ell_2 + \langle \ell_2 \ell_3, w_1 w_2 \rangle \ell_1.$$

1.3 Definition in coordinates

Let $(\xi_0, \dots, \xi_n)$ be the basis of E and its dual basis $(t_0, \dots, t_n)$ in $E^\vee$. Then we can identify $S(E^\vee)$ with polynomial algebra $\mathbb{C}[t_0, \dots, t_n]$ and $S^d(E^\vee)$ with the space $\mathbb{C}[t_0, \dots, t_n]_d$ of homogeneous polynomials of degree d where

$$\langle t_0^{i_0} \cdots t_n^{i_n}, \xi_0^{j_0} \cdots \xi_n^{j_n} \rangle = \begin{cases} i_0! \cdots i_n! & \text{if } (i_0, \dots, i_n) = (j_0, \dots, j_n), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For example, $\langle t_0^2 t_1, \xi_0^2 \xi_1 \rangle = 2$.

1.4 The pairing in terms of differential operators

Since $\langle t_i, \xi_j \rangle = \delta_{ij}$, we can view a basis vector ξ_j as the partial derivative operator $\partial_j = \frac{\partial}{\partial t_j}$. Hence any element $\psi \in S^k(E) = \mathbb{C}[\xi_0, \ldots, \xi_n]_k$ can be viewed as a differential operator

$$D_\psi = \psi(\partial_0, \ldots, \partial_n).$$

Then, the polar pairing becomes

$$\langle \psi(\xi_0, \ldots, \xi_n), f(t_0, \ldots, t_n) \rangle = D_\psi(f).$$

For any monomial $\partial^{\mathbf{i}} = \partial_0^{i_0} \cdots \partial_n^{i_n}$ and any monomial $t^{\mathbf{j}} = t_0^{j_0} \cdots t_n^{j_n}$ we have

$$\partial^{\mathbf{i}}(t^{\mathbf{j}}) = \begin{cases} \dfrac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!}\, t^{\mathbf{j}-\mathbf{i}}, & \text{if } \mathbf{j}-\mathbf{i} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here and later we use the vector notation:

$$\mathbf{i}! = i_0! \cdots i_n!, \qquad \binom{k}{\mathbf{i}} = \frac{k!}{\mathbf{i}!}, \qquad |\mathbf{i}| = i_0 + \cdots + i_n.$$

For example, $\langle \partial_0\partial_1, t_0^2 t_1^2 t_2^2 \rangle = \partial_0\partial_1(t_0^2 t_1^2 t_2^2) = 4 t_0 t_1 t_2^2 = \frac{2!2!2!}{2!} t^{(1,1,2)}$

1.5 The total polarization of f

The total polarization $\tilde{f}$ of a polynomial f is given explicitly by the following formula:

$$\tilde{f}(v_1, \ldots, v_d) = D_{v_1 \cdots v_d}(f) = (D_{v_1} \circ \cdots \circ D_{v_d})(f),$$

where v_k is the direction of the directional derivative, i.e., think of $v = (a_0, \ldots, a_n)$ as a vector in $\mathbb{C}^{n+1}$ with D_v being the differential operator

$$D_v = a_0 \frac{\partial}{\partial t_0} + \cdots + a_n \frac{\partial}{\partial t_n}.$$

Taking $v_1 = \cdots = v_d = v$, we get

$$\tilde{f}(v, \ldots, v) = d! f(v) = D_{v^d}(f) = \sum_{|\mathbf{i}|=d} \binom{d}{\mathbf{i}} a^{\mathbf{i}} \partial^{\mathbf{i}} f.$$

Example 1.1 *Let $f(t) = t_0^2$, $v = (a_0, a_1)$. Then $D_v = a_0\partial_0 + a_1\partial_1$ and*

$$D_{v^2}(f) = D_v(2a_0 t_0) = 2a_0^2 = 2! f(a)$$

Remark 1.1 *The polarization isomorphism was known in the classical literature as the symbolic method. Suppose $f = l^d$ is a d-th power of a linear form. Then*

$$D_v(f) = d\, l(v)\, l^{d-1}$$

and

$$D_{v_1} \circ \cdots \circ D_{v_k}(f) = d(d-1)\cdots(d-k+1)\, l(v_1) \cdots l(v_k)\, l^{d-k}.$$

In classical notation, a linear form $\sum a_i x_i$ on $\mathbb{C}^{n+1}$ is denoted by a_x and the dot-product of two vectors a, b is denoted by (ab). Symbolically, one denotes any homogeneous form by a_x^d and the right-hand side of the previous formula reads as

$$D_{b^k}(a_x) = d(d-1)\cdots(d-k+1)(ab)^k a_x^{d-k}.$$

1.6 A covariant of degree p and order r

Consider the matrix

$$\begin{pmatrix} \xi_0^{(1)} & \cdots & \xi_0^{(d)} & t_0^{(1)} & \cdots & t_0^{(s)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \xi_r^{(1)} & \cdots & \xi_r^{(d)} & t_r^{(1)} & \cdots & t_r^{(s)} \end{pmatrix},$$

where the upper-script mean just repeated basis vector.

The product of k maximal minors such that each of the first d columns occurs exactly k times and each of the last s columns occurs exactly p times represents a *covariant* of degree p and order k.

For example, $(ab)^2 a_x b_x$ represents the *Hessian determinant*

$$\mathrm{He}(f) = \det \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

of a ternary cubic form f.

2. Projective space and varieties

The *projective space* of lines in E will be denoted by $|E|$. We call $\mathbb{P}(E)$ the *dual projective space* of $|E|$. We will often denote it by $|E|^\vee$.

A basis $\xi_0, \ldots, \xi_n$ in E defines an isomorphism $E \cong \mathbb{C}^{n+1}$ and identifies $|E|$ with the projective space

$$\mathbb{P}^n := |\mathbb{C}^{n+1}|.$$

For any nonzero vector $v \in E$ we denote by $[v]$ the corresponding point in $|E|$.

If $E = \mathbb{C}^{n+1}$ and $v = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$, we set

$$[v] = [a_0, \ldots, a_n].$$

We call $[a_0 : \ldots : a_n]$ the *projective coordinates* of a point $[a] \in \mathbb{P}^n$.

Given a homogeneous polynomial $F(x_0, x_1, \ldots, x_n)$ of degree d, the associated projective hypersurface is

$$V(F) = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n \mid F(x_0, x_1, \ldots, x_n) = 0\}.$$

2.1 Polar hypersurfaces

We view $a_0\partial_0 + \cdots + a_n\partial_n \neq 0$ as a point $a \in |E|$ with projective coordinates $[a_0, \ldots, a_n]$.

Definition 2.1 *Let $X = V(f)$ be a hypersurface of degree d in $|E|$ and $x = [v]$ be a point in $|E|$. The hypersurface*

$$P_a^k(X) := V(D_{v^k}(f))$$

of degree $d-k$ is called the k-th polar hypersurface of the point $[a]$ with respect to the hypersurface $V(f)$ (or of the hypersurface with respect to the point).

Example 2.1 *Let $d = 2$, i.e.*

$$f = \sum_{i=0}^{n} \alpha_{ii} t_i^2 + 2 \sum_{0 \le i < j \le n} \alpha_{ij} t_i t_j$$

is a quadratic form on $\mathbb{C}^{n+1}$. For any $x = [a_0, \dots, a_n] \in \mathbb{P}^n$, we have

$$P_x(V(f)) = V(g), \quad g = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i} = 2 \sum_{0 \le i < j \le n} a_i \alpha_{ij} t_j, \qquad \alpha_{ji} = \alpha_{ij}.$$

The linear map $v \mapsto D_v(f)$ is a map from $\mathbb{C}^{n+1}$ to $(\mathbb{C}^{n+1})^\vee$ which can be identified with the polar bilinear form associated to f with matrix $2(\alpha_{ij})$.

2.2 Second definition of polar hypersurface

Let us give another definition of the polar hypersurfaces $P_x^k(X)$. Choose two different points

$$a = [a_0, \dots, a_n], \qquad b = [b_0, \dots, b_n] \in \mathbb{P}^n$$

and consider the line $\ell = ab$ spanned by the two points as the image of the map

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^n, \quad [u_0, u_1] \mapsto u_0 a + u_1 b := [a_0 u_0 + b_0 u_1, \dots, a_n u_0 + b_n u_1].$$

(a parametric equation of ℓ).

The intersection $\ell \cap X$ can be represented by the degree d homogeneous form

$$\varphi^*(f) = f(u_0 a + u_1 b) = f(a_0 u_0 + b_0 u_1, \dots, a_n u_0 + b_n u_1).$$

Using the Taylor formula at $(0,0)$, we can write

$$\varphi^*(f) = \sum_{k+m=d} \frac{1}{k!m!} u_0^k u_1^m A_{km}(a,b),$$

where

$$A_{km}(a,b) = \frac{\partial^d \varphi^*(f)}{\partial u_0^k \, \partial u_1^m}(0,0).$$

Using the Chain Rule, we get

$$A_{km}(a,b) = \sum_{|\mathbf{i}|=k,\, |\mathbf{j}|=m} \binom{k}{\mathbf{i}} \binom{m}{\mathbf{j}} a^{\mathbf{i}} b^{\mathbf{j}} \, \partial^{\mathbf{i}+\mathbf{j}} f = D_{a^k b^m}(f).$$

Observe the symmetry

$$A_{km}(a,b) = A_{mk}(b,a).$$

When we fix a and let b vary in $\mathbb{P}^n$, we obtain a hypersurface

$$V(A(a,x))$$

of degree $d - k$, which is the k-th polar hypersurface of $X = V(f)$ with respect to the point a.

When we fix b and vary a in $\mathbb{P}^n$, we obtain the m-th polar hypersurface

$$V(A(x,b))$$

of X with respect to the point b.

2.3 Reciprocity Theorem

Note that

$$D_{a^k b^m}(f) = D_{a^k}(D_{b^m}(f)) = D_{b^m}(f)(a) = D_{b^m}(D_{a^k}(f)) = D_{a^k}(f)(b).$$

This gives the symmetry property of polars

$$b \in P_a^k(X) \iff a \in P_b^{d-k}(X).$$

Since we are in characteristic 0, if $m \leq d$, $D_{a^m}(f)$ cannot be zero for all a. To see this we use the *Euler formula*:

$$df = \sum_{i=0}^{n} t_i \frac{\partial f}{\partial t_i}.$$

Applying this formula to the partial derivatives, we obtain

$$D_{v^k}(f) = d(d-1)\cdots(d-k+1)f = \sum_{|\mathbf{i}|=k} \binom{k}{\mathbf{i}} t^{\mathbf{i}} \partial^{\mathbf{i}} f.$$

It follows from this formula that, for all $k \leq d$,

$$a \in P_a^k(X) \iff a \in X.$$

This is known as the ***reciprocity theorem***.