Math 220 Final Review Session

definitions and other comments

1 Core definitions

1. Logic and proof language

- A **statement** is a declarative sentence with a definite truth value (true or false).
- An **open sentence** becomes a statement once values are assigned to its variables.
- For an implication $P \implies Q$:
  - **converse:** $Q \implies P$,
  - **contrapositive:** $\neg Q \implies \neg P$,
  - **biconditional:** $P \iff Q$.
- A **direct proof** of $P \implies Q$ starts by assuming P and ends by proving Q.
- A **proof by contrapositive** proves $\neg Q \implies \neg P$ instead.
- A **proof by contradiction** assumes the target statement is false and derives a contradiction.
- A **disproof** of a universal statement is usually a counterexample.

2. Quantifiers and limits

- $\forall x \in A,\ P(x)$ means $P(x)$ is true for every $x \in A$.
- $\exists x \in A$ such that $P(x)$ means $P(x)$ is true for at least one $x \in A$.
- Negations:

$$\neg(\forall x \in A\, P(x)) \iff \exists x \in A \text{ such that } \neg P(x), \qquad \neg(\exists x \in A\, P(x)) \iff \forall x \in A\, \neg P(x).$$

- A sequence (x_n) converges to L if

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N},\ n > N \implies |x_n - L| < \varepsilon.$$

- A function f has limit L as $x \to a$ if

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

3. Induction

- **Mathematical induction:** if $P(1)$ is true and $P(k) \implies P(k+1)$ for all $k \in \mathbb{N}$, then $P(n)$ is true for all $n \in \mathbb{N}$.
- **Strong induction:** assume $P(1), \ldots, P(k)$ and prove $P(k+1)$.

4. Sets and relations

- $A \subseteq B$ means every element of A is also an element of B.
- $A = B$ means $A \subseteq B$ and $B \subseteq A$.
- $\mathcal{P}(A)$ is the power set of A.
- $A \times B = \{(a, b) : a \in A,\ b \in B\}$.
- A relation on A is a subset of $A \times A$.
- A relation is **reflexive**, **symmetric**, **transitive** if it satisfies the standard quantified conditions.
- An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.
- The **equivalence class** of x is

$$[x] = \{y \in A : y \sim x\}.$$

- Congruence modulo n: for $a, b \in \mathbb{Z}$, $a \equiv b \pmod{n}$ means $n \mid (a - b)$.

5. Functions

- A function $f : A \to B$ assigns to each $a \in A$ exactly one element $f(a) \in B$.
- The **domain** is A, the **codomain** is B, and the **range** is $f(A)$.
- For $C \subseteq A$, the **image** is $f(C) = \{f(x) : x \in C\}$.
- For $D \subseteq B$, the **preimage** is $f^{-1}(D) = \{x \in A : f(x) \in D\}$.
- f is **injective** if $f(a_1) = f(a_2) \implies a_1 = a_2$.
- f is **surjective** if every $b \in B$ has some $a \in A$ with $f(a) = b$.
- f is **bijective** if it is both injective and surjective.
- Composition: $(g \circ f)(a) = g(f(a))$.
- An inverse satisfies $g \circ f = i_A$ and $f \circ g = i_B$; a function has an inverse iff it is bijective.

6. Cardinality

- Two sets have the same cardinality if there is a bijection between them.
- A set is **denumerable** if there is a bijection $\mathbb{N} \to A$.
- A set is **countable** if it is finite or denumerable.
- A set is **uncountable** if it is not countable.

2 Common mistakes

- Writing the converse instead of the contrapositive.
- Negating quantified statements incorrectly.
- In induction, assuming what you are trying to prove for $k + 1$.
- In set proofs, proving only one inclusion.
- Confusing codomain with range.
- Confusing preimage with inverse function.
- Claiming a function is onto without solving $f(x) = y$ for an arbitrary y in the codomain.
- Claiming a set is countable without actually producing a bijection or a standard countability argument.
- Using contradiction when a direct or contrapositive proof is much cleaner.

3 Final one-page checklist for students

Before the exam, every student should be able to do the following without notes:

1. State the definitions of subset, relation, equivalence relation, function, injective, surjective, bijective, countable, and uncountable.
2. Negate any quantified statement correctly.
3. Decide whether a statement is best attacked by direct proof, contrapositive, contradiction, cases, or counterexample.
4. Prove a simple induction statement cleanly.
5. Prove two sets are equal by double inclusion.
6. Determine whether a relation is an equivalence relation and compute its classes.
7. Determine whether a function is injective and/or surjective from the definitions.
8. Produce a bijection for standard countability questions.

Source note

This review packet was built from the PLP auxiliary materials for UBC Math 220 together with the weekly worksheets and their old-final-style questions. The problem selection especially emphasizes worksheet themes from weeks 5–12: limits, induction, sets, relations, functions, contradiction, and cardinality.