

Math 220 Final Review Session

Practice problems

1 Warm-up questions

1. What is the difference between a statement and an open sentence?
2. What is the converse of “if n is divisible by 6, then n is divisible by 2”?
3. What is the contrapositive of that statement?
4. Negate: “For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $y > x$.”
5. How do you prove two sets are equal?
6. What is the difference between a relation and an equivalence relation?
7. What is the difference between codomain and range?
8. What is the easiest way to prove a function is not surjective?
9. What is the easiest way to prove two infinite sets have the same cardinality?

On this page, please do problems **A2**, **B1**, **B4**, **C5**, and **D4**.
Solutions 1:

Solutions 2:

2 Main problems

A. Logic, quantifiers, and proof choice

A1. [Medium] Negate each statement and decide whether the original statement is true:

(a) $\forall x \in \mathbb{R} \exists y \in \mathbb{R}$ such that $y + 5 = x$.

(b) $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $x^2 \geq y$.

A2. [Medium] Prove:

$$\forall x, y \in \mathbb{R}, (\forall z > 0, |x - y| < z) \implies (x = y).$$

A3. [Medium] Prove: if n is even, then $n^2 + 4n + 5$ is odd.

A4. [Medium] Prove: if $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

B. Limits and induction

B1. [Medium] Using the definition of convergence, prove that

$$x_n = \frac{n}{n^2 + 1} \rightarrow 0.$$

B2. [Medium] Show that

$$x_n = \frac{n}{\sqrt{n^2 + 1}}$$

does *not* converge to 0.

B3. [Medium] Prove by induction that $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

B4. [Challenge] Prove by strong induction that every natural number greater than 1 has a prime factor.

C. Sets, power sets, relations, congruence

C1. [Medium] Prove or disprove:

$$\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B).$$

C2. [Medium] Prove:

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B).$$

C3. [Challenge] Prove that

$$A \times C \subseteq B \times C \iff A \subseteq B \text{ or } C = \emptyset.$$

C4. [Medium] Let R on \mathbb{Z} be defined by aRb iff $3 \mid (a - b)$. Prove that R is an equivalence relation and describe the distinct equivalence classes.

C5. [Challenge] A relation on $\mathbb{Z} \times \mathbb{N}$ is defined by

$$(x, y) \sim (a, b) \iff xb = ya.$$

Prove that \sim is an equivalence relation and find the equivalence class of $(1, 3)$.

D. Functions

D1. [Medium] Suppose $f : A \rightarrow B$ and $C \subseteq A$. Prove that

$$f(A) - f(C) \subseteq f(A - C).$$

Then find a counterexample to the reverse inclusion.

D2. [Medium] Let $f : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be given by

$$f(n) = (2n + 1, n + 2).$$

Determine whether f is injective and whether it is surjective.

D3. [Medium] Let $f : A \rightarrow B$ be injective. Prove that for all $C_1, C_2 \subseteq A$,

$$f(C_1 \cap C_2) = f(C_1) \cap f(C_2).$$

D4. [Challenge] Prove that the function

$$f : \mathbb{R} \setminus \left\{ \frac{2}{5} \right\} \rightarrow \mathbb{R} \setminus \left\{ -\frac{3}{5} \right\}, \quad f(x) = \frac{3x}{2 - 5x}$$

is bijective.

E. Contradiction and cardinality

E1. [Medium] Prove that $\sqrt{3}$ is irrational.

E2. [Medium] Show that $|\{0, 1\} \times \mathbb{N}| = |\mathbb{Z}|$ by constructing an explicit bijection.

E3. [Medium] Show that $|(0, 1)| = |(0, \infty)|$.

E4. [Challenge] Prove that $|(0, 1]| = |(0, 1)|$ either by constructing a bijection or by using Cantor–Schröder–Bernstein.

E5. [Challenge] Let

$$\mathbb{Z}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}.$$

Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

3 Extra problems

Medium problems

M1. Prove that if $x > 0$, then $x + \frac{2}{x} > 2$.

M2. Prove that if $n \in \mathbb{Z}$, then $n^2 + 3n + 8$ is even.

M3. Prove that if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

M4. Determine whether the relation $aRb \iff 3 \mid (2a - 5b)$ on \mathbb{Z} is reflexive, symmetric, and transitive.

M5. Find a function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ that is injective but not surjective, and another that is surjective but not injective.

M6. Prove that if a sequence converges, then its limit is unique.

Challenging problems

C1. Show that every number $n \in \mathbb{N}$ can be written as $n = 2^k m$ where $k \geq 0$ and m is odd.

C2. Let T be the set of all natural numbers that can be written as a nonnegative integer combination of 3 and 5. Determine $\mathbb{N} \setminus T$.

C3. Let $A = \{a_1, \dots, a_n\}$ be a set of n distinct natural numbers. Prove that some nonempty subset of A has sum divisible by n .

C4. Prove that if $g \circ f$ and $h \circ g$ are bijective, then f, g, h are all bijective.

C5. Prove that the set of polynomials with rational coefficients is countable.